

Dynamical Structure of the Fields in the Light Cone Coordinates

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Abstract

It is well-known that additional constraints emerge in light cone coordinates. We enumerate the number of physical modes in light cone coordinates and compare it with conventional coordinates. We show that the number of Schrödinger modes is divided by two in light cone coordinates. We study the effect of this reduction in the number of ladder operators acting on physical states of a system. We analyse the scalar, spinor and vector field theories carefully to see the effect of changes in the dynamical structure of these theories from the view point of the reduction of Schrödinger modes in light-cone coordinates .

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1 Introduction

Considering the various sub-groups of Poincaré group, in a pioneer paper, Dirac in 1949[1], introduced three forms for relativistic dynamics, instant form (I.F.), front form (F.F.) and point form (P.F.). These forms are related to the various choices of the time axis. The instant form is the usual choice of the coordinate x^0 as the time coordinate, while in front form $(x^0 + x^3)/\sqrt{2}$ is chosen as the time coordinate. The front form has special features with so many applications in theoretical physics, specially in non perturbative *QCD*[2], string theory[3] and so on. In the literature of high energy physics, the front form is recognized with different names such as “*Light Front*”, “*Infinite Momentum Frame*” and “*Light Cone*”. In this paper we use light-cone. For a brief review of light-cone quantization and its application in high energy physics see Ref. [4].

In the light-cone formulation of physical systems, the hyperplane $x^+ = (x^0 + x^3)/\sqrt{2}$ acts as the equal time hyperplane. The light-cone coordinates are $x^\pm = (x^0 \pm x^3)/\sqrt{2} = (x_0 \mp x_3)/\sqrt{2} = x_\mp$ and $x^\perp \equiv (x^1, x^2)$.

For an arbitrary four-vector A^μ with components (A^0, \mathbf{A}) we define the light-cone components as $(A^+, A^i, A^-) = (A^+, \tilde{A})$ where $i = 1, 2$ and $A^\pm = (A^0 \pm A^3)/\sqrt{2}$. So, for invariant space-time length element in Minkowski space we have

$$ds^2 = dx_0^2 - dx_3^2 - dx_i^2 = 2dx_+dx_- - dx_i^2, \quad (1)$$

which shows that the metric has non-diagonal elements.

Historically light-cone coordinates is well-known for particle physicists since it is used to derive some *QCD* sum rules [5][6]. The large variety of applications of light-cone coordinates, come from the advantage of relativistic dynamics of physical systems on the hyperplane of the $x_0 + x_3 = \text{const}$. Dirac mentioned some of these advantages. First, in light-cone coordinates the number of kinematical Poincaré generators are seven while in the conventional formulation only six are kinematical. Second, the non-diagonal form of light-cone metric, enables us to separate the total energy of a system of relativistic particles into center of mass energy and relative energy [4]. This is different from the instant form, in the sense that the appearance of the square root in the relation of energy, $P_0 = (\vec{P}^2 + M^2)^{1/2}$, prohibits a similar separation of variables. These advantages and specially the latter one, have made the light-cone coordinates an appropriate tool for calculating quantities such as wave functions.

One special feature of using light-cone coordinates is emergence of additional constraints compared to the conventional coordinates. We call these additional constraints

light-cone constraints. This change in constraint structure of the theory is well-known[2]. However, the number of light-cone constraints for a generic theory is not well understood yet. Physically we expect no change in the dynamical content of the theory upon changing the coordinates of space-time. So one needs to identify clearly the role of light-cone coordinates on the dynamical behaviour of the system. These are the main task of this work.

We will show explicitly that the light-cone constraints sit in place of half of the physical degrees of freedom. Hence, the number of dynamical degrees of freedom is divided by two, compared to the conventional coordinates. Although this phenomenon is met by physicists working on concrete models[4], it is not clearly recognized as a general rule for an arbitrary model. We will show the light-cone constraints together with the remaining half of the dynamical equations of motion are equivalent to the whole equations of motion in conventional coordinates.

The next problem is how to choose the physical modes to be quantized in light-cone coordinates. For instance, some authors divide the momentum space into two parts and work with, say, the $k_- > 0$ half of the momentum space[12]. This happens when one insists on expanding the fields with the same combination as in conventional coordinates. In this paper we give another approach, in which we maintain the whole momentum space but put away half of the physical modes. In this approach, summation over spin in a spinor field and/or summation over polarization in a gauge field theory is no more necessary in light-cone coordinates. In other words, a light-cone observer is able to observe only one of the spin (polarization) states of a electron (photon).

In the reminder of this paper, we do the above task for the major type of physical theories which are quadratic or first order with respect to velocities. We show that in both types of theories the phenomenon of halving the number of dynamical modes is similar. In section 2, we find the general form of the constraint structure of a theory in light-cone coordinates and enumerate the number of dynamical variables. We do this both for second order and first order Lagrangians. Section 3 denotes to the quantization procedure based on the symplectic approach of quantization, which is more or less a new approach in light-cone quantization. Sections 4 and 5 deal the same procedure for the special case of the spinor field theory and the vector field theory. In section 6 we discuss the case of Yang-Mills theories. The last section denotes out conclusions.

2 Number of dynamical variables

As we mentioned earlier, formulation of theories in light cone coordinates, leads to a different Hamiltonian structure in comparison with conventional coordinates [4]. Since in light-cone coordinates x^+ is the time coordinate, the conjugate momentum is defined as

$$\pi_{F.F.} = \frac{\partial \mathcal{L}}{\partial(\partial_+ \phi)}, \quad (2)$$

which differs from the ordinary instant form momentum $\pi_{I.F} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$ in the sense that

$$\pi_{FF} = \frac{1}{\sqrt{2}}(\pi_{I.F} - \partial_3 \phi). \quad (3)$$

In addition to different Hamiltonian structure, this point leads to a different number of dynamical variables. We investigate the problem in turns for two major important field theoretic systems, i.e. quadratic Lagrangians and first order Lagrangians (with respect to the velocities).

A - Quadratic Lagrangian

Consider a typical theory described by a set of dynamical fields $\phi_a (a = 1, 2, \dots, n)$. Suppose the Lagrangian of the theory is at most quadratic with respect to the partial derivatives of the fields. Taking into account the Lorentz invariance, the most general form of the kinetic term is $g^{ab} \partial_\mu \phi_a \partial^\mu \phi_b$ for some symmetric matrix g . In conventional coordinates (Instant Form) we have $\mathcal{L} = g^{ab}(\partial_0 \phi_a \partial_0 \phi_b - \nabla \phi_a \cdot \nabla \phi_b) + \dots$, and definition of momenta (i.e. $\pi_{I.F}^a \equiv 2g^{ab} \partial_0 \phi_b$) gives no constraint for non singular g . In the light-cone coordinates (Front Form), however, the kinetic term in the Lagrangian is written as $2g^{ab}(\partial_+ \phi_a \partial_- \phi_b - \partial_\perp \phi_a \cdot \partial_\perp \phi_b)$, which gives the conjugate momentum $\pi_{F.F}^a = 2g^{ab} \partial_- \phi_b$. Since there is no velocity in this relation we have the constraints

$$\chi^a \equiv \pi_{I.F}^a - 2g^{ab} \partial_- \phi_b \simeq 0. \quad (4)$$

Hence the non-diagonal form of the light-cone metric changes the constraint structure of the system. If the original theory is not constrained (i.g. Klein-Gordon theory), it will possess some new constraints, while a system which is already a constrained system in conventional coordinates (i.g. Electromagnetism) will possess additional constraints due to the linearity of the Lagrangian with respect to the velocities $\partial_+ \phi_a$.

Suppose there are k first class and m second class constraints on the phase space in conventional coordinates. We also need k subsidiary conditions as gauge fixing conditions to reach the reduced phase space. Hence, there exist all together $2k + m \equiv l$

conditions on the fields in phase space. The number of degrees of freedom is therefore $2n - l$ in Hamiltonian formalism and $n - l/2$ in Lagrangian formalism [10].

Now, by going to the light-cone coordinates the number of remaining degrees of freedom in phase space should be divided by 2. The reason is as follows: the $n - l/2$ physical degrees of freedom correspond to variables in the Lagrangian with truly quadratic terms with respect to the velocities in the conventional coordinates. As we showed, the quadratic terms with respect to conventional velocities (i.e. $(\partial_0\phi_a)^2$) are replaced by terms $\partial_+\phi_a\partial_-\phi_a$ in light-cone coordinates which is linear with respect to velocities. Hence, in the light-cone formulation of the theory we will have $\frac{2n-l}{2}$ additional constraints which we call them "light cone constraints".

The light cone constraints are second class in the sense that their consistency with time determines the corresponding Lagrange multipliers. Since each second class constraint reduces one dynamical variable, we have

$$\mathcal{N}_C^{F.F} = l + \frac{2n-l}{2} = n + \frac{l}{2}. \quad (5)$$

where $\mathcal{N}_C^{F.F}$ is the total number of constraints in front form. In this way half of the dynamical variables of the phase space are omitted by the light cone constraints and the number of degrees of freedom of the theory reduces to $\frac{2n-l}{2}$. In subsequent sections we will see this effect for Klein-Gordon and electromagnetic field theories.

However, note that we have not restricted the physical sector of the theory in phase space by going to the light cone coordinates. To see this we may try to project the additional constraints to the conventional phase space to see if there is any possible reduction. For this reason, we try to transform constraint $\pi_{F.F} - \partial_-\phi = 0$ from light-cone coordinates, to conventional coordinates. By using chain rule for $\pi_{F.F}$ we obtain

$$\pi_{F.F} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \frac{\partial(\partial_0\phi)}{\partial(\partial_+\phi)} + \frac{\partial\mathcal{L}}{\partial(\partial_3\phi)} \frac{\partial(\partial_3\phi)}{\partial(\partial_+\phi)} = \frac{1}{\sqrt{2}}(\partial_0\phi - \partial_3\phi). \quad (6)$$

The right hand side of this equation is the same as $\partial_-\phi$ and so we will get the trivial relation $0 = 0$. So any attempt to find an equivalent hyper-plane for the constraint surface due to the light-cone constraints will lead to trivial equations in the conventional coordinates. In other words, there is no hyper plane in the conventional coordinate phase space equivalent to the hyper plane of light-cone constraints. Therefore the change in constraint structure in light-cone coordinates does not mean that the classical phase space in conventional coordinates is reduced.

B - First Order Lagrangian

The most well known Lagrangian containing a Lorentz invariant first order dynamical term includes $\bar{\psi}_\alpha \gamma_{\alpha\beta}^\mu \partial_\mu \psi_\beta$ ($\alpha, \beta = 1, 2, \dots, n$) as appears in the familiar Dirac Lagrangian. To study the constraint structure of such field theories, we investigate the symplectic matrix of this theory in conventional coordinates as well as light-cone coordinates. By considering $\bar{\psi}_\alpha$ and ψ_β as the independent variables of phase space, in conventional coordinates the dynamical term $\bar{\psi}_\alpha \gamma_{\alpha\beta}^0 \partial_0 \psi_\beta$ gives the symplectic matrix as (see Appendix A)

$$\omega = \begin{pmatrix} 0 & \gamma_{n \times n}^0 \\ -\gamma_{n \times n}^0 & 0 \end{pmatrix}. \quad (7)$$

Since $\det(\gamma^0) \neq 0$, so γ^0 does not have any null eigen-vector. In conventional coordinates, the number of phase space degrees of freedom is $2n$. For example in ordinary Dirac fields, $n = 4$ and the number of phase space variables is 8.

In light-cone coordinates we set $\gamma^\mu \partial_\mu = \gamma^+ \partial_+ + \gamma^- \partial_- + \gamma^\perp \partial_\perp$ where

$$\gamma^\pm = \frac{1}{\sqrt{2}} (\gamma^0 \pm \gamma^3), \quad (8)$$

The dynamical terms in the Lagrangian is $\bar{\psi}_\alpha \gamma_{\alpha\beta}^+ \partial_+ \psi_\beta$ which gives the symplectic matrix as

$$\omega = \begin{pmatrix} 0 & \gamma_{n \times n}^+ \\ -\gamma_{n \times n}^+ & 0 \end{pmatrix}. \quad (9)$$

In 4 dimensional space-time, all representations of Dirac matrices are unitarily equivalent, so it is sufficient to consider a specific representation and investigate the rank of γ^+ . By choosing the chiral representation, we have [15],

$$\gamma^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}. \quad (10)$$

which shows that γ^+ has two null eigen-vectors. Hence, in light-cone coordinates the number of degrees of freedom in the phase space is n instead of $2n$. In the case of 4 dimensional Dirac field it is 4 instead of 8.

Thus, similar to the case of second order Lagrangian, the number of degrees of freedom is divided by two in the light-cone formulation of field theories with first order Lagrangian. Therefore, the number of Schödinger modes in light-cone coordinates are half of those in the conventional coordinates.

3 Quantization procedure

Let us see the effect of change in the constraint structure on the classical dynamics, as well as the quantization procedure of the system. Classically we want to know what happens to half of the degrees of freedom which are absent in the light cone coordinates. In conventional coordinates we need to solve $2n - l$ first order differential equations for the dynamical variables in phase space. However, in light cone coordinates we have $n - l/2$ constraints together with $n - l/2$ first order differential equations with respect to time. Hence, the total number of equations at hand are the same in both formalisms and the physical results are the same, as it should be.

In fact, just the superficial features of the dynamical equations are different in two approaches; i.e. in conventional coordinates all $2n - l$ equations of motion include derivatives with respect to x^0 , while in light cone coordinates $n - l/2$ constraints do not include derivatives with respect to x^+ and the remaining $n - l/2$ do include derivatives with respect to x^+ . A little look at the structure of the light cone constraints in Eq. (4) shows they include in fact, derivatives with respect to ordinary time x^0 upon the change of variables as $\partial_- = (\partial_0 - \partial_3)/\sqrt{2}$.

Our next desire is finding suitable basis for the variables of the reduced phase space in order to follow the dynamics of the system. Suppose we are able to find a suitable basis for the whole phase space of the system, in which imposing the constraints leads to omitting a number of redundant variables. Such a basis is recognized in the literature of constrained systems as the Darbeaux basis [10]. Hence, in comparison with conventional coordinates, the additional light cone constraints lead to a smaller reduced phase space with $n - l/2$ dynamical (x^+ dependent) modes. Then we should solve the equations of motion to find the time dependence of physical modes in terms of $n - l/2$ independent Schrödinger modes (see Appendix A).

In fact, in a Darbeaux basis, in light-cone coordinates, the procedure of solving the dynamics of the system will break into two steps. First, imposing the light-cone second class constraints to omit half of the degrees of freedom; and second, solving the remaining equations of motion to find the Schrödinger modes. Finally one needs to write the expansion of the fields in terms of the Schrödinger modes.

Since the Schrödinger modes play the role of creation and annihilation operators in the quantum theory, one may wonder if different number of Schrödinger modes in light-cone coordinates lead to a different quantum space of physical states. We expect the physical quantities should not depend on the choice of coordinate basis.

To answer this question we should consider the commutation relations of ladder

operators with the Hamiltonian of the system and compare the results in light-cone and conventional coordinates. As a familiar example, we quantize the Klein-Gordon theory in light-cone coordinates using the symplectic method. As we will see, in the light-cone coordinates, according to Eq. (5) we expect one constraint on the classical phase space of the theory which affect the quantization procedure.

Klein-Gordon theory is introduced by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) = \partial_+ \varphi \partial_- \varphi - \frac{1}{2}(\partial_\perp \varphi)^2 - \frac{1}{2}m^2 \varphi^2. \quad (11)$$

The conjugate light-cone momentum is

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_+ \varphi)} = \partial_- \varphi, \quad (12)$$

which introduces the primary constraint $\chi \equiv \pi - \partial_- \varphi \simeq 0$ on the phase space. The total Hamiltonian [9] reads

$$H_T = \int d^3 \tilde{x} \left(\frac{1}{2}(\partial_\perp \varphi)^2 + \frac{1}{2}m^2 \varphi^2 + u(x)\chi(x) \right), \quad (13)$$

where $u(x)$ is the Lagrange multiplier. Assume the equal time fundamental Poisson brackets as

$$\begin{aligned} \{\varphi(\tilde{x}), \varphi(\tilde{x}')\}_{x^+} &= \{\pi(\tilde{x}), \pi(\tilde{x}')\}_{x^+} = 0, \\ \{\varphi(\tilde{x}), \pi(\tilde{x}')\}_{x^+} &= \delta(x^- - x'^-) \delta^2(x^\perp - x'^\perp). \end{aligned} \quad (14)$$

Since the constraint χ considered at different points constitute a system of second class constraints, the consistency condition $\partial_+ \chi(x) = \{\chi(x), H_T\}_{x^+} = 0$ will not give a secondary constraint; instead, it determines the Lagrange multiplier via the equation

$$2\partial_- u(x) = (\partial_\perp \partial_\perp \varphi - m^2 \varphi). \quad (15)$$

To impose the single constraint $\chi(x)$ on the fields, it is more suitable to use the following Fourier expansions

$$\begin{aligned} \varphi &= \frac{1}{(2\pi)^{3/2}} \int d^3 \tilde{k} a(\tilde{k}, x^+) e^{i\tilde{k} \cdot \tilde{x}}, \\ \pi &= \frac{1}{(2\pi)^{3/2}} \int d^3 \tilde{k} c(\tilde{k}, x^+) e^{-i\tilde{k} \cdot \tilde{x}}, \end{aligned} \quad (16)$$

where $\tilde{k} \cdot \tilde{x} \equiv k_- x^- + k_\perp x^\perp$. The physical modes are $a(\tilde{k}, x^+)$ and $c(\tilde{k}, x^+)$. Imposing light-cone constraint $\pi - \partial_- \varphi = 0$ on the expansions (16) gives

$$c(\tilde{k}, x^+) = -ik_- a(-\tilde{k}, x^+). \quad (17)$$

In contrast with conventional coordinates where there are two physical modes, in light-cone coordinates we have only one independent physical mode which we assume to be $a(\tilde{k}, x^+)$. Eq. (17) shows that $c(\tilde{k})$ is determined in terms of $a(\tilde{k})$. Now using Eq. (103), in the appendix A, to construct symplectic two-form, we have

$$\Omega = \int d^3\tilde{k} (-2ik_-) da(-\tilde{k}, x^+) \wedge da(\tilde{k}, x^+). \quad (18)$$

Hence, the Dirac brackets of the physical modes are

$$\left\{ a(\tilde{k}, x^+), a(\tilde{k}', x^+) \right\}_{D.B} = \frac{-1}{2ik_-} \delta^3(\tilde{k} + \tilde{k}'). \quad (19)$$

In terms of the physical modes $a(\tilde{k}, x^+)$, the canonical Hamiltonian is

$$H_c = \int d^3\tilde{k} \frac{1}{2} (\tilde{k}_\perp^2 + m^2) a(\tilde{k}, x^+) a(-\tilde{k}, x^+). \quad (20)$$

Using the canonical Hamiltonian, we are able to write equations of motion of the physical modes as

$$\dot{a}(\tilde{k}, x^+) = \{a, H\} = i\omega_+ a(\tilde{k}, x^+), \quad (21)$$

where

$$\omega_+ \equiv \frac{k_\perp^2 + m^2}{2k_-}. \quad (22)$$

The solution of Eq. (21) is

$$a(\tilde{k}, x^+) = a(\tilde{k}, 0) e^{i\omega_+ x^+}. \quad (23)$$

In contrast with the conventional coordinates where we deal with two coupled first order differential equations of motion, in light-cone coordinates we have only one differential equation. As we mentioned earlier, imposing the light-cone constraint (12), is equivalent to solving one equation of motion of ordinary coordinates. The original fields can be expanded in terms of the Schrödinger modes $a(\tilde{k}, 0)$ as

$$\begin{aligned} \varphi(\tilde{x}, x^+) &= \frac{1}{(2\pi)^{3/2}} \int d^3\tilde{k} a(\tilde{k}, 0) e^{ikx}, \\ \pi(\tilde{x}, x^+) &= \frac{1}{(2\pi)^{3/2}} \int d^3\tilde{k} (-ik_-) a(-\tilde{k}, 0) e^{-ikx}. \end{aligned} \quad (24)$$

By using Eq. (19) and Dirac quantization prescription $\{ \ , \ } \rightarrow -i[\ , \]$, the quantum commutators of the Schrödinger modes as well as original fields can be written

as

$$\left[a(\tilde{k}), a(\tilde{k}') \right] = \frac{-1}{2k_-} \delta^3(\tilde{k} + \tilde{k}'), \quad (25)$$

$$\left[\varphi(\tilde{x}, x^+), \pi(\tilde{y}, x^+) \right] = \frac{1}{2} \delta^3(\tilde{x} - \tilde{y}), \quad (26)$$

$$\left[\varphi(\tilde{x}, x^+), \varphi(\tilde{y}, x^+) \right] = \frac{1}{2} \theta(x^- - y^-) \delta^2(x^\perp - y^\perp). \quad (27)$$

where $\theta(x^- - y^-)$ is the Heaviside step function.

In contrast to conventional coordinates, Eq. (27) shows that the field ϕ does not commute with itself on equal light-cone time hyper plane. Let us investigate this property carefully. We want to find the commutation relation (27) from the non-equal time commutation relation of Klein-Gordon fields in conventional coordinates. In conventional coordinates we have[15]

$$[\varphi(x), \varphi(y)]_{I.F} = \int d^3\mathbf{k} \frac{1}{2\omega_{\mathbf{k}}} (-e^{-ik(x-y)} + e^{ik(x-y)}), \quad (28)$$

which can be written covariantly as

$$[\varphi(x), \varphi(y)]_{I.F} = \int d^4k \delta(k^2 - m^2) \theta(k_0) (-e^{-ik(x-y)} + e^{ik(x-y)}). \quad (29)$$

Note that the subscription I.F in Eq. (29) is no more necessary in covariant form of the commutation relations. Hence, we can transform this integral to light-cone coordinate. By transforming $\delta(k^2 - m^2)$ to light-cone coordinates and integrating over k_+ we have

$$[\varphi(x), \varphi(y)] = \int dk_- d^2k_\perp \frac{1}{2k_-} (-e^{-ik(x-y)} + e^{ik(x-y)}) \theta\left(\frac{k_+ + k_-}{\sqrt{2}}\right) \Big|_{k_+ = \frac{k_\perp^2 + m^2}{2k_-}}.$$

Putting $x^+ = y^+$, we can find the equal light-cone time commutation relations as

$$\left[\varphi(\tilde{x}, x^+), \varphi(\tilde{y}, x^+) \right] = \frac{1}{2} \theta(x^- - y^-) \delta^2(x^\perp - y^\perp), \quad (30)$$

which is exactly the commutation relation (27) we obtained by direct calculation in the light-cone coordinates. This simple result which shows consistency of formulation of Klein-Gordon theory in light-cone and conventional coordinate systems, although expected intuitively, is not shown explicitly in the literature yet. Note that the transformation from light-cone to conventional coordinates is not an ordinary Lorentz transformation.

Now we will turn back to the problem of interpreting different number of ladder operators in light-cone and conventional coordinates. Let see how we can interpret different number of Schrödinger modes in light-cone coordinates?

Consider the commutation relation of ladder operators with the Hamiltonian in both coordinates. In conventional coordinates we have

$$[A(\mathbf{k}), A^\dagger(\mathbf{k})] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}'), \quad (31)$$

$$[H, A(\mathbf{k})] = -\omega_{\mathbf{k}} A(\mathbf{k}), \quad (32)$$

$$[H, A^\dagger(\mathbf{k})] = \omega_{\mathbf{k}} A^\dagger(\mathbf{k}). \quad (33)$$

where $\omega_{\mathbf{k}}$ is the time component of the momentum 4-vector. As we see, the sign of the right hand sides of Eqs. (32) and (33) are different for the annihilation and creation operators. In conventional coordinates, the on shell condition reads $\omega_{\mathbf{k}}^2 = k_0^2 = \mathbf{k}^2 + m^2$; hence, the sign of the spacial components of momentum do not determine the sign of $\omega_{\mathbf{k}}$.

In the light-cone coordinates, however, the number of the ladder operators is divided by 2 and the commutation relations are

$$\begin{aligned} [a(\tilde{k}), a(\tilde{k}')] &= \frac{1}{2k_-} \delta^3(\tilde{k} + \tilde{k}'), \\ [H, a(\tilde{k})] &= -\omega_{\tilde{k}} a(\tilde{k}). \end{aligned} \quad (34)$$

Remembering Eq. (22), shows that the sign of $\omega_{\tilde{k}}$ depends on the sign of k_- . This property divides the momentum space into two parts $k_- > 0$ and $k_- < 0$ where $a(\tilde{k})$ is a creation operator in $k_- > 0$ region and an annihilation operator in $k_- < 0$ region. This point of view differs from conventional approach [16, 17, 18] which insist on introducing two sets of ladder operators for creation and annihilation on the price of restricting the physical domain of momentum coordinate k_- to the region $k_- > 0$.

Let us investigate the effect of light-cone ladder operators on the total momentum of the system. Using the definition of the energy-momentum tensor, $T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \mathcal{L} \delta_\nu^\mu$, the components of momentum in the light-cone coordinates are

$$\begin{aligned} P^+ &= \int d^3 \tilde{x} \pi \partial_- \varphi = \int d^3 \tilde{k} (k_-^2) a(-k) a(k), \\ P^i &= \int d^3 \tilde{x} (-\pi \partial_- \varphi) = - \int d^3 \tilde{k} k_- k_i a(-k) a(k), \quad i = 1, 2 \end{aligned} \quad (35)$$

Using commutation relation (25) we have

$$[P^+, a(\tilde{k})] = [P_-, a(\tilde{k})] = k_- a(\tilde{k}). \quad (36)$$

$$[P_i, a(\tilde{k})] = - [P^i, a(\tilde{k})] = k_i a(\tilde{k}). \quad (37)$$

These relations verifies the interpretation of $a(\tilde{k})$ with $k_- > 0$ ($k_- < 0$) as creation (annihilation) operators.

4 Symplectic light-cone Quantization of Spinor fields

Dirac theory is introduced by the first order Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \quad (38)$$

To quantize this theory in light-cone coordinates, it is convenient to use a decomposition of spinor space by the projection operators[12],

$$\Lambda^\pm = \frac{1}{2}\gamma^\mp\gamma^\pm = \frac{1}{\sqrt{2}}\gamma^0\gamma^\pm, \quad (39)$$

which project the spinor field ψ to $\psi_\pm = \Lambda^\pm\psi$. Using the identities,

$$\gamma^0\gamma^+ = \gamma^-\gamma^0, \quad \Lambda^\pm\Lambda^\mp = 0, \quad (40)$$

we have $\gamma^0\psi_+ = \frac{\sqrt{2}}{2}\gamma^+\psi_+$. Hence, the Lagrangian density of the Dirac field decomposes as,

$$\mathcal{L} = i\sqrt{2}\psi_+^\dagger\partial_+\psi_+ + i\sqrt{2}\psi_-^\dagger\partial_-\psi_- - \psi_-^\dagger(m + i\gamma^i\partial_i)\gamma^0\psi_+ - \psi_+^\dagger(m + i\gamma^i\partial_i)\gamma^0\psi_-. \quad (41)$$

In this way, the Lagrangian (38) can be written as

$$\mathcal{L} = i\sqrt{2}\psi_+^\dagger\partial_+\psi_+ - \psi_-^\dagger\chi_1 - \mathcal{H}_c \quad (42)$$

where the density of canonical Hamiltonian is,

$$\mathcal{H}_c = \frac{1}{\sqrt{2}}\psi_+^\dagger(m + i\gamma^i\partial_i)\gamma^-\psi_-. \quad (43)$$

In the above Lagrangian, the only dynamical variables are ψ_+ and ψ_+^\dagger , while the equations of motion for the variables ψ_-^\dagger and ψ_- give the constraints,

$$\begin{aligned} \chi_1 &\equiv i\partial_-\psi_- - \frac{1}{2}(m + i\gamma^i\partial_i)\gamma^+\psi_+ \approx 0, \\ \chi_2 &\equiv i\partial_-\psi_-^\dagger + \frac{1}{2}(m\psi_+^\dagger - i\partial_i\psi_+^\dagger\gamma^i)\gamma^- \approx 0. \end{aligned} \quad (44)$$

In order to write a suitable mode expansion of the fields ψ_+ and ψ_+^\dagger we look for a complete set of eigenfunctions of the Hamiltonian of the first quantized theory. In conventional coordinates, $u(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$ and $v(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}$ are the eigenfunctions of Dirac Hamiltonian h_D ,

$$h_D = -i\gamma^0\gamma^i\partial_i + m\gamma^0 \quad i = 1, 2, 3, \quad (45)$$

with the energy eigenvalues E_k and $-E_k$ respectively[15]. Actually, the solutions of the eigenvalue equations $h_D\psi(x) = \pm E_k\psi(x)$ can be considered as $u(k)e^{-ik.x}$ and $v(k)e^{ik.x}$ such as,

$$(\gamma^\mu k_\mu - m) u(k) = 0, \quad (46)$$

$$(\gamma^\mu k_\mu + m) v(k) = 0. \quad (47)$$

Each of the equations (46) and (47) have two independent solutions distinguished by the eigenvalues of the component of spin operator, say in the third direction, i.e. Σ^3 . Hence, for every solution of (46) and (47) we can decompose u^1 and u^2 as well as v^1 and v^2 by using the projection operators

$$\mathcal{S}^\mp = \frac{1}{2} (1 \pm \Sigma^3) \quad (48)$$

In this way for the Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ with 8 independent phase space variables, we can set the 8 eigenspinors $\{u_s, v_s, u_s^\dagger, v_s^\dagger\}$ for $s = 1, 2$. So, the summation over spin indices is necessary in the conventional coordinates.

On the other hand, in the light-cone coordinates, due to additional constraints (44), the dimension of the reduced phase space is 4. So, in order to expand independent phase space variables in term of energy eigenfunctions, we need 4 energy eigenfunctions of the Hamiltonian operator. To do this, notice that the Dirac light-cone Hamiltonian operator can be recognized from the canonical Hamiltonian (43) as

$$h_D^{L.C} \equiv \frac{1}{\sqrt{2}} (m + i\gamma^i \partial_i) \gamma^-. \quad (49)$$

Using the plane wave solutions (46) and (47) we can introduce $u_\pm(k) = \Lambda^\pm u(k)$ and $v_\pm(k) = \Lambda^\pm v(k)$. Then it is easy to see,

$$\begin{aligned} h_D^{L.C} u_+(k) &= k_+ u_+(k), \\ h_D^{L.C} u_-(k) &= 0, \\ h_D^{L.C} v_+(k) &= -k_+ v_+(k), \\ h_D^{L.C} v_-(k) &= 0. \end{aligned} \quad (50)$$

As is seen, spinors $u_-(k)$ and $v_-(k)$ are ruled out from the eigenspinors of $h_D^{L.C}$. On the other hand spinors $\{u_+, v_+, u_+^\dagger, v_+^\dagger\}$ form a basis for the four dimensional space of variables ψ_+ and ψ_+^\dagger . In other words, in light-cone coordinates in contrast with conventional coordinates, there is the natural projection operator Λ^\pm for the energy eigenspinors.

In this way there is no need to use the spin projection operators (48) to distinguish the degenerate spinors. Hence, there is no spin summation in expansion of the Dirac fields. Now we are able to expand the dynamical fields ψ_+ and ψ_+^\dagger in the basis $u_+(k)e^{-ik \cdot x}$ and $v_+(k)e^{ik \cdot x}$ and their conjugates as follows,

$$\psi_+(x, x^+) = \int d^3k \left(A(k, x^+) u_+(k) e^{-ik \cdot x} + B^\dagger(k, x^+) v_+(k) e^{ik \cdot x} \right), \quad (51)$$

$$\psi_+^\dagger(x, x^+) = \int d^3k \left(A^\dagger(k, x^+) u_+^\dagger(k) e^{ik \cdot x} + B(k, x^+) v_+^\dagger(k) e^{-ik \cdot x} \right). \quad (52)$$

Before going through the expansion of the fields, let us see what has happened to the state of the eigenstates with spinors $u_+(k)$ and $v_+(k)$. For this reason consider the spin states of energy eigenfunctions in the rest frame $k_r = \frac{1}{\sqrt{2}}(m, 0, 0, m)$. By choosing the rest frame in the relations (46) and (47) we simply have the solutions,

$$u_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \quad (53)$$

Compare these with the conventional basis u^1 and u^2 as v^1 and v^2 in the rest frame as,

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad (54)$$

where $\Sigma^3 u^1 = u^1$, $\Sigma^3 u^2 = -u^2$, $\Sigma^3 v^1 = v^1$ and $\Sigma^3 v^2 = -v^2$. It is easy to see $\Lambda^+ u^1 = \Lambda^+ v^1$ and $\Lambda^+ u^2 = -\Lambda^+ v^2$. This says that the spin states for positive and negative frequency solutions of conventional coordinates are no longer independent after projecting with the operator Λ^+ . Hence, we can recognize the combination of spin states as,

$$\begin{aligned} u_+ &= \Lambda^+(u^1 + u^2) = \Lambda^+(v^1 - v^2), \\ v_+ &= \Lambda^+(v^1 + v^2) = \Lambda^+(u^1 - u^2). \end{aligned} \quad (55)$$

So, energy eigenfunctions u_+ and v_+ are projections of some combinations of spin states. Therefore, Schrödinger modes (or equivalently ladder operators in quantum theory) create and annihilate particles and antiparticles in specific superposition of spin states. This property is in contrast to the quantized Dirac fields in conventional coordinates.

For the dependent fields ψ_- and ψ_-^\dagger , using the constraints (44) we simply have,

$$\psi_-(x, x^+) = \int d^3k \left(A(k, x^+) u_-(k) e^{-ik \cdot x} + B^\dagger(k, x^+) v_-(k) e^{ik \cdot x} \right), \quad (56)$$

$$\psi_-^\dagger(x, x^+) = \int d^3k \left(A^\dagger(k, x^+) u_-^\dagger(k) e^{ik \cdot x} + B(k, x^+) v_-^\dagger(k) e^{-ik \cdot x} \right), \quad (57)$$

where

$$\begin{aligned} u_-(k) &= \frac{m + \gamma^i k_i}{2k_-} \gamma^+ u_+(k), \\ v_-(k) &= \frac{m - \gamma^i k_i}{2k_-} \gamma^+ v_+(k). \end{aligned} \quad (58)$$

These relations are also in consistency with the relations (46) and (47).

To find out the odd Poisson brackets of physical modes, we construct the symplectic two-form $\Omega = \int d^3x i\sqrt{2} d\psi_+^\dagger \wedge d\psi_+$ by using Eqs. (4) and (52) as follows

$$\Omega = \int d^3k i\sqrt{2} \left(dA^\dagger(k, x^+) \wedge dA(k, x^+) - dB^\dagger(k, x^+) \wedge dB(k, x^+) \right). \quad (59)$$

So the odd Poisson brackets of physical modes read

$$\begin{aligned} \{A(k, x^+), A^\dagger(k', x^+)\}_+ &= \frac{-i}{\sqrt{2}} \delta^3(k - k'), \\ \{B(k, x^+), B^\dagger(k', x^+)\}_+ &= \frac{i}{\sqrt{2}} \delta^3(k - k'). \end{aligned} \quad (60)$$

The canonical Hamiltonian (43) in terms of physical modes can be written as,

$$\mathcal{H}_c = \int d^3k \frac{m^2 + k_i^2}{2k_-} \left(A^\dagger(k, x^+) A(k, x^+) - B^\dagger(k, x^+) B(k, x^+) \right). \quad (61)$$

Using the above Hamiltonian and the algebra (60), the equations of motion of physical modes become

$$\begin{aligned} \partial_+ A(k, x^+) &= -i\omega_+ A(k, x^+), \\ \partial_+ A^\dagger(k, x^+) &= i\omega_+ A^\dagger(k, x^+), \\ \partial_+ B(k, x^+) &= -i\omega_+ B(k, x^+), \\ \partial_+ B^\dagger(k, x^+) &= i\omega_+ B^\dagger(k, x^+). \end{aligned} \quad (62)$$

where $\omega_+ = \frac{m^2 + k_i^2}{2k_-}$. By writing the solutions of Eqs. (62) in terms of Schrödinger modes and inserting them into the Eqs. (4) and (52) we have

$$\begin{aligned}\psi_+(x, x^+) &= \int d^3k \left(A(k)u_+(k)e^{-ikx} + B^\dagger(k)v_+(k)e^{ikx} \right), \\ \psi_+^\dagger(x, x^+) &= \int d^3k \left(A^\dagger(k)u_+^\dagger(k)e^{ikx} + B(k)v_+^\dagger(k)e^{-ikx} \right).\end{aligned}\tag{63}$$

Similar results can be written for ψ_- and ψ_-^\dagger where u_- and v_- are derived as in Eqs. (58).

Using $\psi = \psi_+ + \psi_-$ and $u = u_+ + u_-$, the expansions of original Dirac fields becomes,

$$\begin{aligned}\psi(x, x^+) &= \int d^3k \left(A(k)u(k)e^{-ikx} + B^\dagger(k)v(k)e^{ikx} \right), \\ \psi^\dagger(x, x^+) &= \int d^3k \left(A^\dagger(k)u^\dagger(k)e^{ikx} + B(k)v^\dagger(k)e^{-ikx} \right).\end{aligned}\tag{64}$$

As we mentioned earlier, in light-cone coordinates, the summation over spin states is no longer necessary in the expansions of the fields. This property is due to additional constraints (44) which appears in light-cone coordinates. Also same situation arises in light-cone electromagnetic theory where we need not too choose any polarization vector to quantize this theory.

5 Symplectic light-cone Quantization of Vector fields

The familiar electromagnetic theory is a gauge theory with two first class constraints the in conventional coordinates. Let us investigate the constraint structure of this theory in light-cone coordinates.

The Lagrangian $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ of electromagnetic theory should be written in light-cone coordinates as

$$\mathcal{L} = \frac{1}{2}F_{+-}F_{+-} + F_{-i}F_{+i} - \frac{1}{4}(F_{ij})^2.\tag{65}$$

The conjugate momenta are

$$\pi^+ = \frac{\partial \mathcal{L}}{\partial(\partial_+ A_+)} = 0,\tag{66}$$

$$\pi^- = \frac{\partial \mathcal{L}}{\partial(\partial_+ A_-)} = F_{+-},\tag{67}$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_+ A_-)} = F_{-i} \quad i = 1, 2.\tag{68}$$

which give the primary constraints in the light-cone phase space as follows

$$\chi_0 \equiv \pi^+ \simeq 0, \quad (69)$$

$$\chi_i \equiv \pi^i - F_{-i} \simeq 0 \quad i = 1, 2. \quad (70)$$

The total Hamiltonian reads

$$H_T = \int d^3\tilde{x} \left(\frac{1}{2}(\pi^-)^2 + \pi^- \partial_- A_+ + \pi^i \partial_i A_+ + \frac{1}{4} F_{ij} F_{ij} + u(x) \pi^+ + v_i(x) (\pi^i - F_{-i}) \right), \quad (71)$$

where $u(x)$ and $v_i(x)$ are Lagrange multipliers. Assuming the fundamental Poisson brackets as

$$\{A_\mu(\tilde{x}, x^+), \pi^\nu(\tilde{y}, x^+)\} = \delta_\mu^\nu \delta^3(\tilde{x} - \tilde{y}), \quad (72)$$

consistency condition of the constraint χ_0 gives the secondary constraint

$$\phi_0 \equiv \partial_i \pi^i + \partial_- \pi^- \approx 0, \quad (73)$$

while consistency of the constraints χ_i determines the Lagrange multipliers v_i via

$$2\partial_- v_i = \partial_i \pi^- - \partial_j F_{ij}. \quad (74)$$

Consistency of the secondary constraint ϕ does not lead to a new constraint. Hence we have two first class constraints χ_0 and ϕ_0 and two second class constraints χ_i . Comparing with our general discussion on the number of degrees of freedom in section 2, here we have $n = 4$ physical fields A^μ with $k = 2$ first class constraints π_0 and $\partial_i \pi_i (i = 1, 2, 3)$, and no second class constraint in the conventional coordinates. The first class constraints χ_0 and ϕ_0 above are similar to the first class constraints in conventional coordinates. The number of phase space degrees of freedom in conventional coordinates is $2n - 2k = 4$. However, the number of degrees of freedom is divided by two in light cone coordinates due to two additional constraints χ_i which has not any counterpart in conventional coordinates.

To construct the reduced phase space of the system we need two gauge fixing conditions conjugate to our two first class constraints. We begin with the gauge fixing condition $\omega_1 \equiv A_- \approx 0$, which is, in fact, conjugate to the secondary constraint ϕ_0 . The consistency condition of this gauge, i.e. $\partial_+ A_- \approx 0$, gives

$$\omega_2 \equiv \pi^- + \partial_- A_+ \approx 0 \quad (75)$$

which is the second required gauge fixing condition. Consistency of ω_2 gives an equation to determine the Lagrange multiplier $u(x)$. By imposing the 4 constraints and 2 gauge

fixing conditions one obtains a reduced phase space with only two field variables. To determine the smallest set of independent physical modes, we should write a suitable expansion of fields and conjugate momenta and impose these constraint on them. As usual, the Fourier expansion is the suitable one, i.e.

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\tilde{k} e^{i\tilde{k}\cdot\tilde{x}} a_\mu(\tilde{k}, x^+), \quad (76)$$

$$\pi^\mu(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\tilde{k} e^{-i\tilde{k}\cdot\tilde{x}} b^\mu(\tilde{k}, x^+). \quad (77)$$

Imposing the constraints and gauge fixing conditions on the physical modes a_μ and b_μ we find the following six conditions,

$$\begin{cases} b^+(\tilde{k}, x^+) = 0, \\ b^i(\tilde{k}, x^+) = -i \left(k_- a_i(-\tilde{k}, x^+) - k_i a_-(-\tilde{k}, x^+) \right), \\ -ik_- b^-(\tilde{k}, x^+) = ik_i b^i(\tilde{k}, x^+), \\ a_-(\tilde{k}, x^+) = 0, \\ b^-(\tilde{k}, x^+) = ik_- a_+(-\tilde{k}, x^+). \end{cases} \quad (78)$$

There remain two independent physical modes which can be chosen as $a_1(\tilde{k}, x^+)$ and $a_2(\tilde{k}, x^+)$. Here, noticing that the field elements are real functions, we construct a linear superposition of these independent modes in a conjugate way as,

$$a(k, x^+) = a_1(k, x^+) + ia_2(k, x^+) \quad (79)$$

$$a^\dagger(k, x^+) = a_1(-k, x^+) - ia_2(-k, x^+) \quad (80)$$

Rewriting physical modes according to this set of independent modes, we have,

$$\begin{aligned} a_-(k, x^+) &= b^+(k, x^+) = 0 \\ a_+(k, x^+) &= -\left(\frac{k_1 - ik_2}{2k_-}\right)a(k, x^+) - \left(\frac{k_1 + ik_2}{2k_-}\right)a^\dagger(-k, x^+) \\ b^1(k, x^+) &= \frac{-ik_-}{2} (a(-k, x^+) + a^\dagger(k, x^+)) \\ b^2(k, x^+) &= \frac{-ik_-}{2} (a(-k, x^+) - a^\dagger(k, x^+)) \\ b^-(k, x^+) &= \frac{i}{2} ((k_1 - ik_2)a(-k, x^+) + (k_1 + ik_2)a^\dagger(-k, x^+)) \end{aligned} \quad (81)$$

To this end we can construct the symplectic two form as

$$\Omega = \int d^3\tilde{x} 2 (d\pi^\mu \wedge dA_\mu) = \int d^3k (-ik_-) (da^\dagger(k, x^+) \wedge da(k, x^+)) \quad (82)$$

Using the inverse of symplectic matrix (see the appendix) we find the Dirac brackets of physical modes as

$$\{a(k, x^+), a^\dagger(k', x^+)\} = \frac{-1}{ik_-} \delta(k_- - k'_-) \delta^2(k_\perp - k'_\perp). \quad (83)$$

The canonical Hamiltonian in terms of the physical modes can be written as,

$$H_c = \int d^3k \left(\frac{k_1^2 + k_2^2}{2} \right) (a^\dagger(k, x^+) a(k, x^+)). \quad (84)$$

In contrast to conventional coordinates [10], the Hamiltonian (84) is diagonal in terms of the transverse modes $a(\tilde{k}, x^+)$ and $a^\dagger(\tilde{k}, x^+)$. In other words, the transverse modes appear in light cone coordinates in a natural way and we need not to choose any polarization direction to quantize the theory. In fact, by eliminating the redundant modes due to the light cone constraints we need not to assume any polarization direction (as is done for instance in light-cone spinor field where the summation over spin indices has been eliminated).

Using the Hamiltonian of Eq. (84) and the Dirac brackets (83), the equations of motion of physical modes read

$$\partial_+ a(\tilde{k}, x^+) = \{a, H_c\} = i\omega_+ a(\tilde{k}, x^+), \quad (85)$$

$$\partial_+ a^\dagger(\tilde{k}, x^+) = \{a^\dagger, H_c\} = -i\omega_+ a^\dagger(\tilde{k}, x^+), \quad (86)$$

where $\omega_+ = \frac{k_1^2 + k_2^2}{2k_-}$. Inserting the solutions of Eqs. (85) and (86) in the expansions (76) and (77) of fields, we find non vanishing components of electromagnetic fields as,

$$A_+(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k (-1) e^{ikx} \left(\left(\frac{k_1 + ik_2}{2k_-} \right) a(k) - \left(\frac{k_1 + ik_2}{2k_-} \right) a^\dagger(-k) \right), \quad (87)$$

$$A_1(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{ikx} \frac{1}{2} (a(k) + a^\dagger(-k)), \quad (88)$$

$$A_2(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{ikx} \frac{-i}{2} (a(k) - a^\dagger(-k)). \quad (89)$$

where $a(k) \equiv a_i(k, 0)$ are Schrödinger modes. For non vanishing components of momentum fields we have also the following expansions

$$\pi^- = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{-ikx} \frac{i}{2} ((k_1 - ik_2) a(-k) + (k_1 + ik_2) a^\dagger(-k)) \quad (90)$$

$$\pi^1 = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{-ikx} \frac{-ik_-}{2} (a(-k) + a^\dagger(k)) \quad (91)$$

$$\pi^2 = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{-ikx} \frac{-ik_-}{2} (a(-k) - a^\dagger(k)) \quad (92)$$

Using brackets (83) we are able to calculate the Dirac brackets of the fields and conjugate momenta which is in complete agreement with known results[16]. These relations can be seen in Appendix B.

6 Symplectic Light-Cone Quantization of Yang-Mills Theories

Yang-Mills theories are theories for describing the behaviour of elementary gauge particles intermediating the physical interactions given by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a, \quad (93)$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{bc}^a A_\mu^b A_\nu^c$, in which g is the coupling constant, and f_{ab}^c are the structure constants of a Lie group (i.e. the gauge group). The index a runs over $1, 2, \dots, N$ where N is the number of generators of the gauge group.

In conventional coordinates, this theory includes N first class primary constraints and N secondary first class constraints. Taking into account $2N$ gauge fixing conditions, there remain $8N - 4N$ degrees of freedom[10]. However, in light cone coordinates according to Eq.(5), we expect to have $2N$ additional second class constraints which is half number of dynamical degrees of freedom.

The Lagrangian density in light-cone coordinate reads

$$\mathcal{L} = \frac{1}{2} F_{+-}^a F_{+-}^a + F_{-i}^a F_{+i}^a - \frac{1}{4} (F_{ij}^a)^2. \quad (94)$$

The conjugate momentums are similar to Eqs. (66) - (68) with the additional subscript a on the momentum fields π_a^μ conjugate to the fields A_μ^a . Again $\phi_a^0 \equiv \pi_a^+ \simeq 0$ and $\phi_a^i \equiv \pi_a^i - F_{-i}^a \simeq 0$ are primary constraints and the total Hamiltonian reads

$$H_T = \int d^3\tilde{x} \left(\frac{1}{2} (\pi_a^-)^2 + \pi_a^- (D_-)^{ab} A_+^b + \pi_a^i (D_i)^{ab} A_+^b + \frac{1}{4} F_{ij}^a F_{ij}^a + u^d(x) \pi_d^+ + v_i^e(x) (\pi_e^i - F_{-i}^e) \right), \quad (95)$$

where $u^d(x)$ and $v_i^e(x)$ are Lagrange multipliers and $(D_\nu)^{ab} \equiv \delta_b^a \partial_\nu - g f_{bc}^a A_\nu^c$ is the covariant derivative. Assuming the fundamental Poisson brackets as

$$\{A_\mu^a(\tilde{x}, x^+), \pi_b^\nu(\tilde{y}, x^+)\} = \delta_\mu^\nu \delta_b^a \delta^3(x - y), \quad (96)$$

the consistency condition $\partial_+ \phi_a^0 \approx 0$ gives a set of secondary first class constraints as

$$\chi^a \equiv (D_i)^{ab} \pi_b^i + (D_-)^{ab} \pi_b^- \approx 0. \quad (97)$$

Consistency condition of this secondary constraints holds identically. Consistency of the constraints ϕ_a^i determines the Lagrange multipliers v_i^e via the relations

$$2(D_-)^{ab} v_i^b + (D_j)^{ab} F_{ij}^b - (D_i)^{ab} \pi_a^- = 0 \quad (98)$$

To construct the reduced phase space of the system we need $2N$ gauge fixing conditions conjugate to our $2N$ first class constraints ϕ_a^0 and χ^a . To choose required gauge fixing conditions we simply generalize the Electromagnetic gauge fixing conditions and choose $\omega_1^a \equiv A_-^a \approx 0$ conjugate to ϕ_a^0 . The consistency condition of ω_1^a gives another gauge fixing condition as

$$\omega_2^a \equiv \pi_a^- + \partial_- A_+^a \approx 0 \quad (99)$$

Hence, there are altogether $6N$ conditions on the fields and conjugate momenta as

$$\left\{ \begin{array}{l} \phi_a^0 \equiv \pi_a^+ \approx 0, \\ \phi_a^i \equiv \pi_a^i - F_{-i}^a \approx 0 \quad i = 1, 2, \\ \chi^a \equiv (D_i)^{ab} \pi_b^i + (D_-)^{ab} \pi_b^- \approx 0, \\ \omega_1^a \equiv A_-^a \approx 0, \\ \omega_2^a \equiv \pi_a^- + \partial_- A_+^a \approx 0. \end{array} \right. \quad (100)$$

In the scalar theory and Electromagnetic field, we simply choose Fourier expansions of fields to find the independent physical modes. But in the non-Abelian Yang-Mills theories, there are some non-linear terms in constraints such as

$$\phi_a^i = \pi_a^i - F_{-i}^a = \pi_a^i - \partial_\mu A_\nu^a + \partial_\nu A_\mu^a - g f_{bc}^a A_\mu^b A_\nu^c = 0 \quad (101)$$

Due to the existence of non-linear terms such $g f_{bc}^a A_\mu^b A_\nu^c$ we are not able to impose this constraint on the Fourier expansion of fields to construct the reduced phase space. However, this problem is not the problem of light-cone quantization and is the fundamental problem of quantization of non-Abelian Yang-Mills theories. To quantize this theory, one can set the limit $g = 0$ and quantize the theory perturbatively but attempt to quantize the theory directly fails due to lack of an appropriate expansion of fields which we enables us impose the constraints.

7 Conclusion

The appearance of additional constraints on a field theory in light-cone coordinates is well-known. However, for the expansions of fields and conjugate momenta in light-cone coordinates according to physical modes, authors use the usual expansions of conventional coordinates with an extra Heaviside step function. This step function divides the momentum space into two parts which enables us to choose say $k_- > 0$ part of the momentum space. In this paper, we have proposed alternative expansions of the fields and conjugate momenta on the whole momentum space for the scalar, fermionic and vector fields. Our expansion is based on the fact that the number of independent physical modes must be equal to the number of degrees of freedom in phase space. To do this, we exactly have investigated the dynamical structure of phase space variables by enumerating the number of degrees of freedom as well as independent physical modes as follows.

First of all, we have shown that non-diagonal form of the light-cone metric causes changes in the constraint structure of the field theories described by the quadratic and first order Lagrangians. We showed exactly that half of the dynamical equations of motion are replaced by the light-cone constraints, hence the number of dynamical degrees of freedom is divided by two, comparing with the conventional coordinates. Although this phenomenon is met by physicists working on concrete models[4], it is not clearly recognized as a general role for an arbitrary model. We showed that the light-cone constraints together with the remaining half of the dynamical equations of motion are equivalent to the whole equations of motion in conventional coordinates.

Second, since the number of independent physical modes are equal to the number of degrees of freedom, using the symplectic method of quantization, we chose the most appropriate set of independent physical modes to expand the phase space variables. By imposing the constraints on the phase space variables to obtain the reduced phase space, we priori have solved half of the equations of motion. By solving the remaining equations of motion, we obtained Schrödinger modes. Then we showed that each one of Schrödinger modes can play the role of creation or annihilation operator depending on the sign of the k_- component of the momentum vector.

At the end, using symplectic method of quantization and analysing the dynamical structure of the phase space variables, we have proposed alternative expansions of the fields in the whole momentum space with true number of physical modes. Notice that the number of independent physical modes must be equal to the number of independent phase space variables. In the case of scalar field, we have obtained relations (24) which

shows that only one set of Schrödinger modes i.e. $a(k)$ act as ladder operators. In the cases of fermionic and vector fields, our expansions have more significant remarks.

In the case of fermionic fields, we have obtained relations (64) for phase space variables. Additional light-cone constraints eliminate summation over spin indices in these expansions. We showed that ladder operators in the quantum theory create and annihilate particles and antiparticles in a specific superposition of spin indices.

Similar situation arises in the case of the vector field. As illustrated in the relations (87,88,89), the summation over polarization states is no longer necessary in the expansions of the fields and conjugate momenta.

We also investigated the constraint structure of the Non-Abelian Yang-Mills theories in light-cone coordinates; and showed that the number of degrees of freedom is again half of those of conventional coordinates. Due to the existence of non-linear terms such $g f_{bc}^a A_\mu^b A_\nu^c$ in the expressions of the constraints, we are not able to impose the constraints on the Fourier expansion of the fields to construct the reduced phase space. However, this problem is not due to light-cone quantization and is the fundamental problem of quantization of non-Abelian Yang-Mills theories. To quantize this theory, one can set the limit $g = 0$ and quantize the theory perturbatively. However, any attempt to quantize the theory directly fails due to lack of appropriate expansions of the fields which enable us to impose the constraints.

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Appendix A

Symplectic Method of Quantization

In this appendix, we briefly review the symplectic method of quantization which is proposed originally by Faddeev and Jackiw [8] and is showed that is equivalent to Dirac's method of quantization of constrained systems[7]. For a comprehensive review of this method see Ref. [13].

For practical purpose, the symplectic method is based on the following steps:

- A) First, we should determine the complete constraint structure of systems[9]. This means that we should determine the primary constraints by calculating conjugate momenta. Then by applying the consistency conditions and defining fundamental Poisson brackets of canonical variables, we should obtain the secondary constraints. In this step we also need to construct the canonical Hamiltonian of the system.

After investigating the constraint structure of the system, we should classify constraints as first and second class constraints using fundamental Poisson brackets. According to a conjecture by Dirac[9], first class constraints are the generators of gauge transformations. In this way, we should fix the gauges by imposing additional gauge fixing conditions on the system. In Ref.[11], the essential requirements for an appropriate gauge fixing conditions are given.

- B) In the second step, we should propose appropriate expansions of the fields and which enables us to impose constraints and gauge fixing conditions to get the reduced phase space. This step determines physical modes as the smallest set of time dependent variables which uniquely describe every state of the classical system. In many familiar cases the Fourier expansion is an appropriate choice.
- C) This step is the most important part of the quantization procedure in which we find the canonical commutation relation of fields. The symplectic two form is defined as[13]

$$\Omega = \frac{1}{2} \int d^3x \sum_i d\pi^i \wedge d\phi_i \quad (102)$$

Where ϕ_i and π^i are fields and conjugate momentum fields respectively. By imposing constraints and gauge fixing conditions on the field expansions, we obtain them in terms of the physical modes. Using the expansions of the fields in terms

of a set of physical modes, normally gives

$$\Omega = \sum_{i,j} \int d^3k \, \omega_{ij} da_i(k, t) \wedge da_j(k, t) \quad (103)$$

where a_i are physical modes and ω_{ij} is the symplectic matrix. Finally by inverting the symplectic matrix, we get the Dirac brackets of the physical modes

$$\{a_i(k, t), a_j(k', t)\}_{D.B} = \omega^{ij} \delta^3(k - k') \quad (104)$$

where $\omega^{ij} \omega_{jk} = \delta_k^i$.

D) In order to consider the dynamics of the theory we should construct the canonical Hamiltonian of the system. Then we should write it in terms of physical modes derived in step(B). By solving the equations of motion of the physical modes based on the Dirac brackets in Eq.(104), one can write the physical modes in terms of certain quantities at a given time, for example $a_i(k, 0)$. These quantities are called Schrödinger modes. The basic Algebra of Dirac brackets, i.e. Eq.(104), can be written in terms of Schrödinger modes.

E) Finally, using Dirac prescription of quantization, we quantize the theory according to

$$\{ \ , \ } \rightarrow -i [\ , \]. \quad (105)$$

Note that by converting Schrödinger modes to operators, we will have the expansion of the fields in terms of creation and annihilation operators.

Appendix B

Dirac Brackets of Vector Fields

Using brackets (83) we are able to calculate the Dirac brackets of the fields as

$$\{A_+(\tilde{x}, x^+), \pi^-(\tilde{y}, x^+)\} = \frac{i}{2}|x^- - y^-|\theta(x^- - y^-)\partial_\perp\partial_\perp\delta^2(x^\perp - y^\perp) \quad (106)$$

$$\{A_+(\tilde{x}, x^+), \pi^i(\tilde{y}, x^+)\} = \frac{i}{2}|x^- - y^-|\theta(x^- - y^-)\partial_i\partial_i\delta^2(x^\perp - y^\perp) \quad (107)$$

$$\{A_+(\tilde{x}, x^+), A_+(\tilde{y}, x^+)\} = \frac{1}{2}|x^- - y^-|^2\theta(x^- - y^-)\partial_\perp\partial_\perp\delta^2(x^\perp - y^\perp) \quad (108)$$

$$\{A_+(\tilde{x}, x^+), A_i(\tilde{y}, x^+)\} = \frac{i}{2}|x^- - y^-|\theta(x^- - y^-)\partial_i\delta^2(x^\perp - y^\perp) \quad (109)$$

$$\{A_i(\tilde{x}, x^+), A_j(\tilde{y}, x^+)\} = \frac{i}{2}\delta_i^j\theta(x^- - y^-)\delta^2(x^\perp - y^\perp) \quad (110)$$

$$\{A_i(\tilde{x}, x^+), \pi^-(\tilde{y}, x^+)\} = \frac{1}{2}\theta(x^- - y^-)\partial_i\delta^2(x^\perp - y^\perp) \quad (111)$$

$$\{A_i(\tilde{x}, x^+), \pi^j(\tilde{y}, x^+)\} = \frac{-1}{2}\delta_i^j\theta(x^- - y^-)\partial_i\delta^2(x^\perp - y^\perp) \quad (112)$$

$$\{\pi^-(\tilde{x}, x^+), \pi^-(\tilde{y}, x^+)\} = \frac{i}{2}\theta(x^- - y^-)\partial_\perp\partial_\perp\delta^2(x^\perp - y^\perp) \quad (113)$$

$$\{\pi^-(\tilde{x}, x^+), \pi^i(\tilde{y}, x^+)\} = \frac{-i}{2}\theta(x^- - y^-)\partial_i\delta^2(x^\perp - y^\perp) \quad (114)$$

$$\{\pi^i(\tilde{x}, x^+), \pi^j(\tilde{y}, x^+)\} = \frac{i}{2}\delta_i^j\theta(x^- - y^-)\partial_i\delta^2(x^\perp - y^\perp) \quad (115)$$

By transforming $\{ \ , \} \rightarrow -i[\ , \]$ according to Dirac prescription of quantization, we finally achieve the quantized electromagnetic theory in light-cone coordinates.

References

- [1] Dirac, P. A. M., *Forms of relativistic dynamics*, *Rev. Mod. Phys.* 21(1949)392.
- [2] Brodsky SJ, Pauli HC, Pinsky SS. *Quantum chromodynamics and other field theories on the light cone*. *Physics Reports*. (1998) Aug 1;301(4):299-486.
- [3] Green, Michael B., and John H. Schwarz. *Covariant description of superstrings*. *Physics Letters B* 136.5 (1984): 367-370..
- [4] Heinzl, T. *Light-cone quantization: Foundations and applications*. In *Methods of Quantization*, (2001), (pp. 55-142). Springer Berlin Heidelberg.
- [5] Fubini, S. and Furlan, G. *Renormalization effects for partially conserved currents*, (1965), Universit degli Studi, Fac. di Scienze.
- [6] Dashen, R. and Gell-Mann, M., *Representation of local current algebra at infinite momentum*. *Physical Review Letters*, (1966), 17(6), p.340.
- [7] Shirzad, A., and M. Mojiri. *The difficulty of symplectic analysis with second class systems*. *Journal of mathematical physics* 46.1 (2005): 012702.
- [8] Faddeev, L., and R. Jackiw. *Hamiltonian reduction of unconstrained and constrained systems*. *Physical Review Letters* 60.17 (1988): 1692.
- [9] P. A. M. Dirac, *Lectures on Quantum Mechanics* New York: Yeshiva University Press, (1964).
- [10] Henneaux, Marc, and Claudio Teitelboim. *Quantization of gauge systems*, Princeton university press, (1992). Sundermeyer. K, *Constrained Dynamics*, Volume 169 of Lecture notes in physics, Springer-Verlag, (1982).
- [11] Shirzad, A. *Full and partial gauge fixing*. *Journal of Mathematical Physics* 48.8 (2007): 082303.
- [12] Kogut, John B., and Davison E. Soper. *Quantum electrodynamics in the infinite-momentum frame*, *Physical Review D* 1.10 (1970): 2901.
- [13] Shirzad, A., A. Bakhshi, and Y. Koohsarian. *Symplectic quantization of massive bosonic string in background B-field*. *Modern Physics Letters A* 27.13 (2012): 1250073.

- [14] Huszar, M. *Light front quantization by Dirac's method. Journal of Physics A: Mathematical and General* 9.8 (1976): 1359.
- [15] Peskin, Michael, and Dan Schroeder. *An introduction to quantum field theory.* (1995).
- [16] Srivastava, Prem P. *Constraints and Hamiltonian in light-front quantized field theory. Il Nuovo Cimento A* (1965-1970) 107.4 (1994): 549-557.
- [17] Burkardt, Matthias. *Light front quantization. Advances in Nuclear Physics.* Springer US, 2002. 1-74.
- [18] Popov, Alexey V. *Symplectic approach to lightcone QCD. Nuclear Physics A* 836.1 (2010): 136-158.